# Examen – Probabilités – 2ème session

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Jeudi 24 Janvier 2008, 14h-16h

Good luck !

## 1 Coins

Suppose each of three persons tosses a coin. If the outcome of one of the tosses differs from the other outcomes, then the game ends. If not, then the persons start over and retoss their coins. Assuming fair coins, what is the probability that the game will end with the first round of tosses ? If all three coins are biased and have probability $\frac{1}{4}$ of landing heads, what is the probability that the game will end at the first round ?

## 2 Balls

An urn contains $b$ black balls and $r$ red balls. One of the balls is drawn at random, but when it is put back in the urn $c$ additional balls of the same color are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $\frac{b}{b+r+c}$.

## 3 More coins

A fair coin is independently flipped $n$ times, $k$ times by $A$ and $n-k$ times by $B$. Show that the probability that $A$ and $B$ flip the same numbers of heads is equal to the probability that there are a total of $k$ heads.

## 4 Trials

Consider three trials, each of which is a success or not. Let $X$ denote the number of successes. Suppose that $E[X] = 1.8$.

($a$) What is the largest possible value of $P[X = 3]$ ?

($b$) What is the smallest possible value of $P[X = 3]$ ?

In both cases, construct a probability scenario that results in $P[X = 3]$ having the desired value.

## 5 Successive draws

A coin having probability $p$ of coming up heads is successively flipped until two of the most recent three flips are heads. Let $N$ denote the number of flips (note that if the first two flips are heads then $N = 2$). Find the expectation $E(N)$.

# 6 Markov

1. Consider a regular Markov chain $(X_0, X_1, \dots)$ with $M+1$ states (numbered from 0 to $M$) and transition matrix $P$. Suppose that the sum over each column of $P$ equals 1 (that is $\sum_i P_{ij} = 1$ for all $j$). Show that the limiting probability vector $\pi$ is given by

$$\pi_j = \frac{1}{M+1}, \qquad j = 0, 1, \dots, M$$

2. A particle moves on a circle through points that have been marked $0, 1, 2, 3, 4$ (in a clockwise order). At each step it has probability $p$ of moving to the right (clockwise) and $1-p$ to the left (counterclockwise). Let $X_n$ denote its location on the circle after $n$ steps. Find the transition matrix $P$ and limiting probability vector $\pi$ of this Markov chain.

# 7 Markov ?

Consider a regular Markov chain $(X_0, X_1, \dots)$ with limiting probability vector $\pi$. Define the process $(Y_1, Y_2, \dots)$ by $Y_n = (X_{n-1}, X_n)$. That is, $Y_n$ keeps tracks of the last two states of the original chain. Is $(Y_1, Y_2, \dots)$ a Markov chain ? If so, determine its transition matrix and find $\lim_{n \to +\infty} P[Y_n = (i, j)]$.